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Minimal Complete Classes of Invariant Tests for Equality of Normal Covariance Matrices and Sphericity

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The problem of testing equality of two normal covariance matrices, $\Sigma_1 = \Sigma_2$ is studied. Two alternative hypotheses, $\Sigma_1 \neq \Sigma_2$ and $\Sigma_1 - \Sigma_2 > 0$ are considered. Minimal complete classes among the class of invariant tests are found. The group of transformations leaving the problems invariant is the group of nonsingular matrices. The maximal invariant statistic is the ordered characteristic roots of $S_1 S_2^{-1}$, where S_i, $i = 1, 2$, are the sample covariance matrices. Several tests based on the largest and smallest roots are proven to be inadmissible. Other tests are examined for admissibility in the class of invariant tests. The problem of testing for sphericity of a normal covariance matrix is also studied. Again a minimal complete class of invariant tests is found. The popular tests are again examined for admissibility and inadmissibility in the class of invariant tests.

INTRODUCTION AND SUMMARY

The problems of testing equality of two normal covariance matrices and testing sphericity of a normal covariance matrix are classical problems in multivariate analysis. See, for example, Anderson [1, Chap. 10] and Muirhead [7, Chap. 8]. In this paper we consider the admissibility of invariant tests in these common testing problems. Two problems (two-sided and one-sided cases) are based on S_1 and S_2, independent, where

$$S_1 \sim W_p(n_1, \Sigma_1) \quad \text{and} \quad S_2 \sim W_p(n_2, \Sigma_2), \tag{1.1}$$

and $W_p(n, \Sigma)$ is the Wishart distribution on $p \times p$ matrices with n degrees

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of freedom and expectation $n\Sigma$. We assume that $p \geqslant 2$, $n_1 \geqslant p$, and $n_2 \geqslant p$, and that Σ_1 and Σ_2 are positive definite. We consider testing

$$H_0\colon \Sigma_1 = \Sigma_2 \qquad \text{versus} \qquad H_A\colon \Sigma_1 \neq \Sigma_2, \tag{1.2}$$

and

$$H_0\colon \Sigma_1 = \Sigma_2 \qquad \text{versus} \qquad H_A\colon \Sigma_1 > \Sigma_2, \tag{1.3}$$

where $\Sigma_1 > \Sigma_2$ means that $\Sigma_1 - \Sigma_2$ is positive definite.

The third problem tests for sphericity of a covariance matrix. That is, we have

$$S \sim W_p(n, \Sigma), \tag{1.4}$$

$n \geqslant p \geqslant 2$, $\Sigma > 0$, and test

$$H_0\colon \Sigma = \sigma^2 I \qquad \text{versus} \qquad H_A\colon \Sigma \neq \sigma^2 I, \tag{1.5}$$

where $\sigma^2 > 0$ is unspecified and I is the $p \times p$ identity matrix.

Problems (1.2) and (1.3) are invariant under the group $Gl(p)$ of $p \times p$ nonsingular matrices, which acts on (S_1, S_2) via

$$A\colon (S_1, S_2) \to (AS_1A', AS_2A') \tag{1.6}$$

for $A \in Gl(p)$, and on (Σ_1, Σ_2), similarly. A maximal invariant statistic and parameter are respectively

$$z = \operatorname{diag}\{\text{ordered characteristic roots of } S_1 S_2^{-1}\},$$

and

$$\alpha = \operatorname{diag}\{\text{ordered characteristic roots of } \Sigma_1 \Sigma_2^{-1}\}.$$

See Anderson [1, Theorem 10.6.1]. However, to develop our results it is more convenient to work with the maximal invariants x and θ, where $x_i = (z_i - 1)/(z_i + 1)$ and $\theta_i = (1 - \alpha_{p-i+1})/(1 + \alpha_{p-i+1})$. As such,

$$x = \operatorname{diag}\{\text{ordered characteristic roots of } (S_1 - S_2)(S_1 + S_2)^{-1}\}, \tag{1.7}$$

and

$$\theta = \operatorname{diag}\{\text{ordered characteristic roots of } (\Sigma_2 - \Sigma_1)(\Sigma_1 + \Sigma_2)^{-1}\}.$$

Hence, $x \in \mathscr{D}(p)$, the set of $p \times p$ diagonal matrices, and the diagonal elements of x satisfy $1 \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_p \geqslant -1$. The invariance-reduced problem (1.2) then tests

$$H_0\colon \theta = 0 \qquad \text{versus} \qquad H_A\colon \theta \in \Theta - \{0\}, \tag{1.8}$$

where

$$\Theta = \{\theta \in \mathscr{D}(p) \mid 1 > \theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_p > -1\}, \tag{1.9}$$

based on x with sample space

$$\mathscr{X} = \{x \in \mathscr{D}(p) \mid 1 > x_1 > x_2 > \cdots > x_p > -1\}. \tag{1.10}$$

Note that we have eliminated from the sample space the set of measure zero on which the x_i's are not distinct. A popular test for (1.2), in terms of x, is likelihood ratio test (LRT), which rejects H_0 when

$$|I + x|^{-n_1/2}\, |I - x|^{-n_2/2} > c. \tag{1.11}$$

Another test, which arises from our complete class rejects H_0 when

$$\frac{n_1+n_2}{2}(\operatorname{tr} x)^2 + \operatorname{tr} x^2 > c, \qquad 0 < c < \frac{n_1+n_2}{2} p^2 + p. \tag{1.12}$$

(In each case, the constant c is chosen to provide the desired level.) Other tests, including those based on tr x and the extreme characteristic roots, are listed in Muirhead [7, p. 332]. One such rejects H_0 when

$$\operatorname{tr} x < c_1 \qquad \text{or} \qquad \operatorname{tr} x > c_2, \qquad -p < c_1 < c_2 < p. \tag{1.13}$$

Tests based on the extreme roots of $S_1 S_2^{-1}$, which are equivalent to those based on the extreme roots of $(S_1 - S_2)(S_1 + S_2)^{-1}$ include those which reject H_0 when

$$x_1 < c_1 \qquad \text{or} \qquad x_1 > c_2; \tag{1.14}$$

$$x_p < c_1 \qquad \text{or} \qquad x_p > c_2; \tag{1.15}$$

$$x_p < c_1 \qquad \text{and} \qquad x_1 > c_2; \tag{1.16}$$

and

$$x_p < c_1 \qquad \text{or} \qquad x_1 > c_2. \tag{1.17}$$

In each case, $-1 < c_1 < c_2 < 1$.

Maximal invariants for problem (1.3) are x and θ as in (1.7), but now the alternative parameter space is smaller:

$$H_0\colon \theta = 0 \qquad \text{versus} \qquad H_A\colon \theta \in \Theta^+ - \{0\}, \tag{1.18}$$

where

$$\Theta^+ = \{\theta \in \mathscr{D}(p) \mid 0 > \theta_1 > \theta_2 > \cdots > \theta_p > -1\}. \tag{1.19}$$

The LRT for problem (1.3) modifies (1.11) by using the statistic $\bar{x}$ instead of x, where $\bar{x} \in \mathscr{D}(p)$ is defined by

$$\bar{x}_i = \max\left\{x_i, \frac{n_1 - n_2}{n_1 + n_2}\right\}. \tag{1.20}$$

The test rejects H_0 when

$$|I + \bar{x}|^{-n_1/2}\, |I - \bar{x}|^{-n_2/2} > c, \qquad c > 0. \tag{1.21}$$

The locally best invariant test rejects H_0 when

$$\operatorname{tr} x > c, \tag{1.22}$$

where $-p < c < p$ (see Giri [4]). The extreme root tests have rejection regions

$$x_1 > c \tag{1.23}$$

and

$$x_p > c, \tag{1.24}$$

where $-1 < c < 1$.

The following theorem summarizes our admissibility/inadmissibility results for problems (1.8) and (1.18).

THEOREM 1.1. (a) *The LRT* (1.11) *when* $n_1 > 2(p-1)$ *and* $n_2 > 2(p-1)$, *and the test* (1.12), *are admissible in the invariant problem* (1.8). *The tests* (1.13)–(1.17) *are inadmissible.* (b) *The test* (1.22) *is admissible in the invariant problem* (1.18). *The LRT* (1.21) *and root tests* (1.23) *and* (1.24) *are inadmissible.*

The result for the test (1.22) follows from the essential uniqueness of its local properties, although it is also easy to prove its admissibility by using Theorem 3.1. The admissibility of the LRT (1.11) in problem (1.8) follows from the stronger result of Kiefer and Schwartz [6] which proves the LRT is admissible Bayes for the original problem (1.2).

The inadmissibility results are all based on violation of the following convexity property. (We represent a test as a measurable function $\phi: x \to [0, 1]$, where $\phi(x)$ is the probability of rejecting H_0 when x is observed.)

PROPERTY 1.2. *The test* ϕ *equals* $1 - I_A$, *a.e.* $[\nu]$, *for some convex set* $A \subseteq X$ *for which no three points of the boundary in* X *are collinear.*

Here, ν is the measure on X when $\theta = 0$, which is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^p$, and I_A is the indicator function of A. We will prove the next proposition in Sections 2 and 3.

PROPOSITION 1.3. (a) *A necessary condition for a test ϕ to be admissible for problem* (1.8) *is that it equal* $1 - I_A$, *a.e.* $[v]$, *where A is either of the form* $\{x \mid \operatorname{tr} x \leqslant a\}$, *or* $\{x \mid \operatorname{tr} x \geqslant b\}$, *or ϕ satisfy Property* 1.2.

(b) *A necessary condition for a test ϕ to be admissible for problem* (1.18) *is that it equal* $1 - I_A$, *a.e.* $[v]$, *where A is of the form* $\{x \mid \operatorname{tr} x \leqslant a\}$, *or ϕ satisfy Property* 1.2.

It is fairly easy to see that tests (1.13)–(1.17), (1.21), (1.23), and (1.24) are not of the form required by Proposition 1.3.

Now turn to problem (1.5). The invariance group for this problem is the direct product $(0, \infty) \times O(p)$, where the operation for $(0, \infty)$ is multiplication and $O(p)$ is the group of $p \times p$ orthogonal matrices. The action is

$$(c, \Gamma)\colon S \to c\Gamma S\Gamma'. \tag{1.25}$$

A maximal invariant statistic and parameter are, respectively,

$$y = \operatorname{diag}\{\text{ordered characteristic roots of } S/\operatorname{tr} S\} \tag{1.26}$$

and

$$\lambda = \operatorname{diag}\{\text{ordered characteristic roots of } \Sigma/\operatorname{tr} \Sigma\}. \tag{1.27}$$

We prefer to use the parameter

$$\omega = p\lambda - I, \tag{1.28}$$

so that the hypotheses in (1.5) become

$$H_0\colon \omega = 0 \qquad \text{versus} \qquad H_A\colon \omega \in \Omega - \{0\}, \tag{1.29}$$

where

$$\Omega = \{\omega \in \mathscr{D}(p) \mid (p-1) > \omega_1 \geqslant \cdots \geqslant \omega_p > -1 \text{ and } \operatorname{tr} \omega = 0\}. \tag{1.30}$$

The LRT for problem (1.5) rejects H_0 when

$$|y| < c, \qquad 0 < c < 1, \tag{1.31}$$

where $|y|$ is the determinant of y. The locally most powerful invariant test has rejection region

$$S_y^2 \equiv \frac{1}{p}\, \Sigma(y_i - \bar{y})^2 > d, \tag{1.32}$$

where $\bar{y} = \Sigma y_i/p = 1/p$. See Sugiura [8]. Relevant root tests have rejection regions

$$y_1 > a, \tag{1.33}$$

$$y_p < b, \tag{1.34}$$

$$y_1 > a \quad \text{and} \quad y_p < b, \tag{1.35}$$

and

$$y_1 > a \quad \text{or} \quad y_p < b, \tag{1.36}$$

where $a \in (1/p, 1)$ and $b \in (0, 1/p)$.

THEOREM 1.4. *The LRT* (1.31) *and LMPI* (1.32) *test are admissible for problem* (1.29). *The root tests* (1.32), (1.34), (1.35), *and* (1.36) *are inadmissible if $p \geq 3$. When $p = 2$, the uniformly most powerful invariant test has rejection region $\{y \mid y_1 > c\}$, $c \in (\frac{1}{2}, 1)$.*

Again the admissibility of the LRT is found in Kiefer and Schwartz [6], and that for the LMPI test is due to its uniqueness. See also Theorem 3.1. The inadmissibility results follow from the next proposition.

PROPOSITION 1.5. *A necessary condition for a test ϕ to be admissible for problem* (1.29) *when $p \geq 3$ is that it satisfy Property* 1.2 (*with $\mathscr{Y}$, the space of y, in place of $\mathscr{X}$.*)

The proof of this proposition and the $p = 2$ result are given in Section 3.

Our main results in the paper are Theorems 2.1, 2.2, and 3.1, which contain the minimal complete classes of tests for the reduced problems (1.8), (1.18), and (1.29). The proofs are in Section 4.

2. TESTING $\Sigma_1 = \Sigma_2$

We will use Brown and Marden [2] heavily, so that our first task is to find the likelihood ratio for x. Recall

$$z = \text{diag}\{\text{ordered characteristic roots of } S_1 S_2^{-1}\}, \tag{2.1}$$

and

$$\alpha = \text{diag}\{\text{ordered characteristic roots of } \Sigma_1 \Sigma_2^{-1}\}.$$

Then from James [5, Eqs. (33) and (65)], we have that

$$f_\alpha(z)/f_I(z) = |\alpha|^{-n_1/2}\,|I+z|^{\beta} \int_{O(p)} |I+z\Gamma\alpha^{-1}\Gamma'|^{-\beta}\,\rho(d\Gamma),$$
$$\beta = (n_1+n_2)/2, \tag{2.2}$$

where $f_\alpha(z)$ is the density of z when α obtains, and ρ is the Haar probability measure on $O(p)$. Now by (1.7) and (2.1)

$$z = (I+x)(I-x)^{-1} \quad \text{and} \quad \dot{\alpha} = (I-\theta)(I+\theta)^{-1}, \tag{2.3}$$

where $\dot{\alpha} = \operatorname{diag}(\alpha_p, ..., \alpha_1)$. Thus the ratio (2.2) in terms of (x, θ) is

$$|I+\theta|^{n_1/2}\,|I-\theta|^{n_2/2} \int_{O(p)} |I+x\Gamma\theta\Gamma'|^{-\beta}\,\rho(d\Gamma). \tag{2.4}$$

(To see this, note that α can be replaced by $\dot{\alpha}$ in (2.2),

$$|\alpha| = |I-\theta|\;|I+\theta|^{-1},$$
$$|I+z| = |I+(I+x)(I-x)^{-1}| = |I-x|^{-1}\,|2I| = |I-x|^{-1}\,2^p,$$

and

$$\begin{aligned}
|I+z\Gamma\alpha^{-1}\Gamma'| &= |I+(I-x)^{-1}\,(I+x)\,\Gamma(I+\theta)(I-\theta)^{-1}\,\Gamma'|\\
&= |I-x|^{-1}\,|\Gamma'(I-x)\,\Gamma+\Gamma'(I+x)\,\Gamma(I+\theta)(I-\theta)^{-1}|\\
&= |I-x|^{-1}\,|I-\theta|^{-1}\,|\Gamma'(I-x)\,\Gamma(I-\theta)+\Gamma'(I+x)\,\Gamma(I+\theta)\\
&= |I-x|^{-1}\,|I-\theta|^{-1}\,|2I+2\Gamma'x\Gamma\theta|\\
&= |I-x|^{-1}\,|I-\theta|^{-1}\,|I+x\Gamma\theta\Gamma'|\,2^p.)
\end{aligned}$$

Let $a(\theta) = |I+\theta|^{-n_1/2}\,|I-\theta|^{-n_2/2}$, and define $R_\theta(x)$ to be $a(\theta)$ times the quantity in (2.4), so that

$$R_\theta(x) = \int_{O(p)} |I+x\Gamma\theta\Gamma'|^{-\beta}\,\rho(d\Gamma). \tag{2.5}$$

To define the minimal complete classes, we need the derivatives

$$l(x) = (l_1(x), ..., l_p(x))', \qquad \text{where } l_i(x) = \frac{\partial}{\partial\theta_i} R_\theta(x)\,|_{\theta=0}, \tag{2.6}$$

ad

$$V(x) = \{V_{ij}(x)\}_{i,j=1}^{p}, \qquad \text{where } V_{ij}(x) = \frac{\partial^2}{\partial\theta_i\,\partial\theta_j} R_\theta(x)\,|_{\theta=0}. \tag{2.7}$$

For $\mu \in \mathbb{R}^p$, $M_0 \in \mathscr{S}(p)$ (the set of nonnegative definite symmetric $p \times p$ matrices), $H \in \mathscr{F}(\bar{\Theta} - \{0\})$, where $\mathscr{F}(\Psi)$ is the set of nonnegative measures on Ψ and $\bar{\Theta}$ is the closure of Θ in $\mathscr{D}(p)$, and $c \in \mathbb{R}$, define

$$\begin{aligned} d(x) &\equiv d(x; \mu, M_0, H, c) \\ &= \mu' l(x) + \tfrac{1}{2} \operatorname{tr} M_0 V(x) + \int_{\bar{\Theta} - \{0\}} \frac{R_\theta(x) - 1 - \boldsymbol{\theta}' l(x)}{\|\theta\|^2} H(d\theta) - c, \end{aligned} \tag{2.8}$$

where $\boldsymbol{\theta}$ is the vector $(\theta_1, ..., \theta_p)$. We have extended the domain of $R_\theta(x)$ to $\bar{\Theta} x \mathscr{X}$ by continuity.

For problem (1.8) define Φ to be the class of all tests of the form

$$\phi(x) = \begin{cases} 1 & \text{if } d(x; \mu, M_0, H, c) > 0 \\ 0 & \text{if } d(x; \mu, M_0, H, c) < 0, \text{ a.e. } [\nu], \end{cases} \tag{2.9}$$

for some

$$(\mu, M_0, H, c) \in C(\Theta) \times \{\gamma J \mid \gamma \geqslant 0\} \times \mathscr{F}_0(\bar{\Theta} - \{0\}) \times \mathbb{R} - \{(0, 0, 0, 0)\}, \tag{2.10}$$

where $C(\Theta)$ is the smallest convex cone containing Θ,

$$C(\Theta) = \{\theta \in \mathscr{D}(p) \mid \theta_1 \geqslant \theta_2 \geqslant \cdots \geqslant \theta_p\}, \tag{2.11}$$

J is the $p \times p$ matrix consisting of all ones, and $\mathscr{F}_0(\bar{\Theta} - \{0\})$ is the set of measures $G \in \mathscr{F}(\bar{\Theta} - \{0\})$ which satisfy

$$\int_{\bar{\Theta} - \{0\}} \frac{\theta_i - \theta_{i+1}}{\|\theta\|^2} G(d\theta) < \infty, \qquad i = 1, ..., p-1. \tag{2.12}$$

THEOREM 2.1. *The class Φ is minimal complete for problem* (1.8).

The proof will be given in Section 4.

Now we look at the local terms (2.6) and (2.7) more closely. From James [5, Eqs. (13) and (33)], we see that $R_\theta(x)$ in (2.5) is a generalized hypergeometric function of two matrix arguments with zonal polynomial expansion:

$$R_\theta(x) = {}_1F_0(\beta; -\theta, x) = \sum_{k=0}^{\infty} \sum_{\kappa \in \mathscr{P}(k)} \frac{c_\kappa}{k!} \frac{C_\kappa(-\theta)\, C_\kappa(x)}{C_\kappa(I)}. \tag{2.13}$$

Here, $\mathscr{P}(k)$ is the set of partitions of the integer k, and for each partition κ, $C_\kappa(\cdot)$ is the corresponding zonal polynomial and c_κ is a positive constant.

The zonal polynomials for $k \leqslant 6$ are given in the Appendix of James [5]. We need the $k \leqslant 2$ terms,

$$R_\theta(x) = 1 - \frac{\beta}{p} \operatorname{tr} \theta \operatorname{tr} x + \frac{1}{6} \frac{\beta(\beta+1)}{p(p+2)} [(\operatorname{tr} x)^2 + 2 \operatorname{tr} x^2][(\operatorname{tr} \theta)^2 + 2 \operatorname{tr} \theta^2]$$

$$+ \frac{1}{3} \frac{\beta(\beta - 1/2)}{p(p-1)} [(\operatorname{tr} x)^2 - \operatorname{tr} x^2][(\operatorname{tr} \theta)^2 - \operatorname{tr} \theta^2)] + h_\theta(x), \quad (2.14)$$

where

$$h_\theta(x) = \sum_{k=3}^{\infty} \sum_{\kappa \in \mathscr{P}(k)} \frac{c_\kappa}{k!} \frac{C_\kappa(-\theta)\, C_\kappa(x)}{C_\kappa(I)}. \quad (2.15)$$

Since for $\kappa \in \mathscr{P}(k)$ and $A \in \mathscr{D}(p)$, $C_\kappa(A)$ is a symmetric polynomial in $A_1, ..., A_p$ of degree k, and each monomial making up the polynomial has a nonnegative coefficient (see Farrell [3, Problem 13.1.13]), we can derive that

$$|C_\kappa(x)| \leqslant C_\kappa(I) \qquad \text{for } x \in \mathscr{X}, \quad (2.16)$$

since $|x_i| < 1$ for each i, and that for any $\varepsilon \in (0, 1)$, and $k \geqslant 3$,

$$\left| \frac{C_\kappa(-\theta)}{\|\theta\|^2} \right| \leqslant \varepsilon^{k-2} C_\kappa(I)$$

$$\leqslant \varepsilon^{(5/6)k-2} C_\kappa(\varepsilon^{1/6} I) \qquad \text{for } \|\theta\| \leqslant \varepsilon. \quad (2.17)$$

Thus, since $h_\theta(x)$ in (2.15) is a sum of terms with $k \geqslant 3$,

$$\sup_{x \in \mathscr{X}} \sup_{\|\theta\| \leqslant \varepsilon} \left| \frac{h_\theta(x)}{\|\theta\|^2} \right| \leqslant \sum_{k=3}^{\infty} \sum_{\kappa \in \mathscr{P}(k)} \frac{c_\kappa}{k!} C_\kappa(\varepsilon^{1/6} I)\, \varepsilon^{(5/6)k-2}$$

$$\leqslant \varepsilon^{1/2} \sum_{k=0}^{\infty} \sum_{\kappa \in \mathscr{P}(k)} \frac{c_\kappa}{k!} C_\kappa(\varepsilon^{1/6} I)$$

$$= \varepsilon^{1/2} |I - \varepsilon^{1/6} I|^{-\beta}$$

$$= \varepsilon^{1/2} (1 - \varepsilon^{1/6})^{-\beta p}. \quad (2.18)$$

Hence (2.14) and (2.18) make it easy to show that from (2.6) and (2.7),

$$l_i(x) = -\frac{\beta}{p} \operatorname{tr} x, \qquad i = 1, ..., p, \quad (2.19)$$

and

$$V_{ij}(x)=\begin{cases}\dfrac{\beta(\beta+1)}{p(p+2)}\,[(\operatorname{tr} x)^2+2\operatorname{tr} x^2] & \text{if } i=j\\[2ex] \dfrac{1}{3}\dfrac{\beta(\beta+1)}{p(p+2)}\,[(\operatorname{tr} x)^2+2\operatorname{tr} x^2] & \\ \qquad+\dfrac{2}{3}\dfrac{\beta(\beta-1/2)}{p(p-1)}\,[(\operatorname{tr} x)^2-\operatorname{tr} x^2] & \text{if } i\neq j.\end{cases} \tag{2.20}$$

Hence if we take μ and M_0 as in (2.10),

$$\mu' l(x) = -\frac{\beta}{p}\,(\Sigma\mu_i)\operatorname{tr} x \equiv \delta \operatorname{tr} x \tag{2.21}$$

and

$$\operatorname{tr} M_0 V(x) = \gamma \sum_i \sum_j V_{ij}(x) = \gamma\beta[\beta(\operatorname{tr} x)^2 + \operatorname{tr} x^2], \tag{2.22}$$

where $\delta \in \mathbb{R}$ and $\gamma \geqslant 0$. Thus we can alternatively define Φ to consist of all tests of the form

$$\phi(x)=\begin{cases}1 & \text{if } \bar{d}(x;\delta,\gamma,H,c)>0\\ 0 & \text{if } \bar{d}(x;\delta,\gamma,H,c)<0, \text{ a.e. } [\nu],\end{cases} \tag{2.23}$$

for

$$(\delta,\gamma,H,c)\in\mathbb{R}\times[0,\infty)\times\mathscr{F}_0(\bar{\Theta}-\{0\})\times\mathbb{R}-\{(0,0,0,0)\}, \tag{2.24}$$

where

$$\bar{d}(x;\delta,\gamma,H,c)=\delta\operatorname{tr} x+\gamma[\beta\operatorname{tr} x)^2+\operatorname{tr} x^2] + \int_{\bar{\Theta}-\{0\}} \frac{(R_\theta(x)-1+(\beta/p)\operatorname{tr}\theta\operatorname{tr} x)}{\|\theta\|^2}\,H(d\theta)-c. \tag{2.25}$$

We turn to Theorem 1.1(a) The test (1.12) is easily seen to be in Φ, hence is admissible for problem (1.8), by taking $(\delta,\gamma,H,c)=(0,1,0,c)$ in (2.23). The remainder of the theorem follows as in the Introduction pending proof of Proposition 1.3(a), which we now give.

Proof of Proposition 1.3.a: We start by showing that $R_\theta(x)$ is strictly convex in x if $\theta\neq 0$. Using the representation of (2.2) obtainable from Wijsman [9], we write

$$\frac{f_\alpha(z)}{f_0(z)}=\frac{|\alpha|^{n_2/2}\int |AA'|^{\beta-p/2}\,e^{-(1/2)\operatorname{tr} AS_1A'}e^{-(1/2)\operatorname{tr}\alpha AS_2A'}\,dA}{\int |AA'|^{\beta-p/2}\,e^{-(1/2)\operatorname{tr} A(S_1+S_2)A'}\,dA}, \tag{2.26}$$

where the integrals are over $A \in \mathcal{G}l(p)$. Manipulations familiar in such situations yield the ratio in terms of (x, θ) to be

$$K\,|I+\theta|^{-n_1/2}\,|I-\theta|^{-n_2/2} \int |AA'|^{\beta-p/2}\, e^{-(1/2)\operatorname{tr} AA'} e^{-(1/2)\operatorname{tr}\theta AxA'}\, dA, \tag{2.27}$$

where K is a positive constant. It is then possible to prove that if $\theta \neq 0$, the expression in (2.27) is strictly convex in x, hence $R_\theta(x)$ is strictly convex in x.

Now consider a test $\phi \in \Phi$ and the corresponding set from (2.23),

$$B \equiv \{x \mid \bar{d}(x; \delta, \gamma, H, c) \leqslant 0\}. \tag{2.28}$$

Since $R_\theta(x)$ is strictly convex in x if $\theta \neq 0$, $\operatorname{tr} x$ is convex in x, and $\beta(\operatorname{tr} x)^2 + \operatorname{tr} x^2$ is strictly convex in x, we have by (2.25) that

$$\begin{array}{lll} \text{(i)} & \bar{d}(x; \delta, \gamma, H, c) \text{ is strictly convex in } x \text{ if } (\gamma, H) \neq (0, 0); & (2.29) \\ \text{(ii)} & \bar{d}(x; \delta, \gamma, H, c) \equiv -c \quad \text{for } c \neq 0 \text{ if } (\delta, \gamma, H) = (0, 0, 0); & \\ \text{(iii)} & \bar{d}(x; \delta, \gamma, H, c) = \delta \operatorname{tr} x - c \quad \text{for } \delta \neq 0 \text{ otherwise.} & \end{array}$$

In any of the cases in (2.29), B of (2.28) is convex, and since $\bar{d}$ is continuous in x and ν is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^p$, the boundary of B in $\mathscr{X}$ equals $\{x \mid \bar{d}(x) = 0\}$ and has ν-measure zero. Hence $\phi = 1 - I_A$, a.e. $[\nu]$.

If case (ii) or (iii) in (2.29) holds, then B is either $\{x \mid \operatorname{tr} x \leqslant \alpha\}$, or $\{x \mid \operatorname{tr} x \geqslant b\}$, where we take a or $b \in [-p, p]$. (In case (ii), B is either empty or $\mathscr{X}$, so we take $a = -p$ or $a = p$, for example.) If case (ii) holds, then since the boundary of B is $\{x \mid \bar{d}(x) = 0\}$, and $\bar{d}$ is strictly convex, no three points on the boundary of B can be collinear, i.e., Property 1.2 holds. Hence Proposition 1.3(a) is proven.

Now turn to the one-sided problem (1.18). Define the class of tests Φ^+, which is a subset of Φ, to consist of all tests of the form

$$\phi(x) = \begin{cases} 1 & \text{if } d^+(x; \delta, H, c) > 0 \\ 0 & \text{if } d^+(x; \delta, H, c) < 0, \text{ a.e. } [\nu], \end{cases} \tag{2.30}$$

for

$$(\delta, H, c) \in [0, \infty) \times \mathscr{F}(\bar{\Theta}^+ - \{0\}) \times \mathbb{R} - \{(0, 0, 0)\}, \tag{2.31}$$

where

$$d^+(x; \delta, H, c) = \delta \operatorname{tr} x + \int_{\bar{\Theta}^+ - \{0\}} \frac{(R_\theta(x) - 1)}{\|\theta\|^2} H(d\theta) - c. \tag{2.32}$$

The function $R_\theta(x)$ is given in (2.5). The proof of the next result is in Section 4.

THEOREM 2.2. *The class Φ^+ is minimal complete for problem* (1.18).

The proof of Proposition 1.3(b) follows as the proof of part (a) above, where we note that $\delta \geqslant 0$. An additional result is available. Note that, from (2.27),

$$-\operatorname{tr} \theta A x A' = -\sum_i \sum_j \theta_i x_j a_{ij}^2. \tag{2.33}$$

Since for $\theta \in \Theta^+$, $\theta_i \leqslant 0$ for each i, the expression in (2.33) is nondecreasing in each x_i, hence $R_\theta(x)$ in (2.27) is nondecreasing in each x_i. It is easy to extend the definition of $R_\theta(x)$ to

$$x \in \mathscr{X}^* \equiv \{x \in \mathscr{D}(p) \mid -1 < x_i < 1 \text{ for each } i\}.$$

This new $R_\theta(x)$ and the corresponding $d^+(x)$ are invariant under permutations of the elements of x. See (2.27) which is in terms of ordered x_i's. Together with the convexity of d^+, we have by Proposition 4.C.2d of Marshall and Olkin [10] that d^+ satisfies the weak submajorization monotonicity property, i.e.,

$$\text{If } x, y \in \mathscr{X} \text{ with } x \leqslant y_1, x_1 + x_2 \leqslant y_1 + y_2, ..., x_1 + \cdots + x_p \leqslant y_1 + \cdots + y_p,$$
$$\text{then } d^+(x) \leqslant d^+(y). \tag{2.34}$$

Thus we have the following:

PROPOSITION 2.3. *A necessary condition for a test ϕ to be admissible for problem* (1.18) *is that it equal $1 - I_B$, a.e. $[v]$, for some set B which is monotone nonincreasing in the ordering* (2.34).

3. TESTING SPHERICITY

Let $g_\lambda(y)$ be the density of Y in (1.26) when λ in (1.27) obtains. From Sugiura [8, Eq. (1.3)], we have that

$$\frac{g_\lambda(y)}{g_I(y)} = |\lambda|^{n/2} \int_{O(p)} (\operatorname{tr} y\Gamma\lambda\Gamma')^{-\tau} \rho(d\Gamma), \qquad \tau = np/2. \tag{3.1}$$

Recall from Section 2 that ρ is the Haar probability measure on $O(p)$.

Rewriting the ratio (3.1) in terms of ω of (1.28), and multiplying it by $|I+\omega|^{-n/2}$, yields

$$R_\omega^*(y) \equiv \int_{O(p)} (1+\operatorname{tr} y\Gamma\omega\Gamma')^{-\tau}\, \rho(d\Gamma). \tag{3.2}$$

(Recall that $\operatorname{tr} y = 1$.)

We need to find the derivatives corresponding to (2.6) and (2.7). Note that for $|a| \leqslant 1$,

$$(1+a)^{-\tau} = 1 - \tau a + \frac{\tau(\tau+1)}{2} a^2 + o(a^2), \tag{3.3}$$

where $o(a^2)$ is as $a \to 0$, uniformly in $|a| \leqslant \varepsilon$ for any $\varepsilon \in (0, 1)$. Since $y_i \in (0, 1)$ for each i,

$$(\operatorname{tr} y\Gamma\omega\Gamma')^2 \leqslant (\Sigma\, |\omega_i|^2) \leqslant p\, \|\omega\|^2. \tag{3.4}$$

Hence from (3.2) and (3.3) we have

$$R_\omega^*(y) = 1 - \tau \int_{O(p)} (\operatorname{tr} y\Gamma\omega\Gamma')\, \rho(d\Gamma) + \frac{\tau(\tau+1)}{2} \int_{O(p)} (\operatorname{tr} y\Gamma\omega\Gamma')^2\, \rho(d\Gamma)$$
$$+ o(\|\omega\|^2), \tag{3.5}$$

where $o(\|\omega\|^2)$ is as $\omega \to 0$, uniformly in $y \in \mathscr{Y}$. Using zonal polynomials as in Sugiura [8], or calculating directly, we obtain

$$\int (\operatorname{tr} y\Gamma\omega\Gamma')\, \rho(d\Gamma) = \frac{\operatorname{tr} y \operatorname{tr} \omega}{p} = 0 \qquad (\text{since } \operatorname{tr} \omega = 0) \tag{3.6}$$

and

$$\int (\operatorname{tr} y\Gamma\omega\Gamma')^2\, \rho(d\Gamma) = \frac{2\, \|\theta\|^2}{p(p+2)(p-1)} [p \operatorname{tr} y^2 - 1]$$
$$= \frac{2\, \|\theta\|^2}{(p+2)(p-1)} S_y^2. \tag{3.7}$$

See (1.32). Thus (3.5), (3.6), and (3.7) show that

$$l_i^*(y) \equiv \frac{\partial}{\partial \omega_i} R_\omega^*(y)\, |_{\omega=0} = 0 \tag{3.8}$$

and

$$V_{ij}^*(y) \equiv \frac{\partial^2}{\partial \omega_i\, \partial \omega_j} R_\omega^*(y)\, |_{\omega=0} = \frac{4}{(p+2)(p-1)} S_y^2 I_{\{i=j\}}. \tag{3.9}$$

Now let Φ^* be the class of tests of the form

$$\phi(y)=\begin{cases}1 & \text{if } d^+(y;\gamma,H,c)>0\\ 0 & \text{if } d^+(y;\gamma,H,c)<0, \text{ a.e. } [\nu^*],\end{cases} \tag{3.10}$$

for

$$(\delta,H,c)\in[0,\infty)\times\mathscr{F}(\bar{\Omega}-\{0\})\times\mathbb{R}-\{(0,0,0)\}, \tag{3.11}$$

where

$$d^*(y)\equiv d^*(y;\gamma,H,c)=\gamma S_y^2+\int_{\bar{\Omega}-\{0\}}\frac{R_\omega^*(y)-1}{\|\omega\|^2}H(d\omega)-c \tag{3.12}$$

and ν^* is the null measure on $\mathscr{Y}$. It is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{p-1}$.

THEOREM 3.1. *The class Φ^* in minimal complete for problem* (1.29).

The proof is indicated in Section 4.

Proposition 1.5 is proved as Proposition 1.3, where we note that S_y^2 and $R_\omega^*(y)$ for $\omega\neq 0$ are strictly convex in y. The latter result follows from the facts that $(1+a)^{-\tau}$ is strictly convex in a and $\operatorname{tr} y\Gamma\omega\Gamma'$ is linear in the diagonal elements of y and, with ρ probability one the coefficients multiplying each diagonal element of y are nonzero.

Finally, consider the case $p=2$ in Theorem 1.4. Extend the definition of $R_\omega^*(y)$ to the set $\{y\in\mathbb{R}^2 \mid y_1+y_2=1,\ y_1>0,\ y_2>0\}$. Note that $R_\omega^*(y)$ is invariant under the permutation of y_1 and y_2, and S_y^2 and $R_\omega^*(y)$ when $\omega\neq 0$ are strictly convex in y. Thus d^* is also permutation invariant and strictly convex unless $(\delta,H)=(0,0)$. Thus d^* has a minimum at $(y_1,y_2)=(\frac{1}{2},\frac{1}{2})$ and is either constant or strictly increasing as y_1 moves away from $\frac{1}{2}$. Thus the only admissible tests are those with acceptance regions essentially of the form $\{y \mid |y_1\leqslant c|,\ c\in[\frac{1}{2},1]$.

4. PROOFS OF THEOREMS 2.1, 2.2, AND 3.1

In this section we will refer to Brown and Marden [2] by B – M. We first use B – M Theorem 2.4 to prove the classes Φ, Φ^+, and Φ^* essentially complete for their respective problems (1.8), (1.18), and (1.29). We need to verify B – M Assumptions 2.1, 2.2, and 2.3.

Start with problem (1.8). Assumption 2.1 requires that for each x, $R_\theta(x)$ as a function on $\bar{\Theta}$ satisfies

$$0<R_\theta(x)<\infty \qquad \text{for } \theta\in\bar{\Theta}. \tag{4.1}$$

By inspection of (2.5), $R_\theta(x)$ is positive. By (2.16) with $-\theta$ instead of x we have that

$$|C_\kappa(-\theta)| \leqslant C_\kappa(I),$$

hence by (2.13) and (2.5)

$$R_\theta(x) \leqslant \prod_{i=1}^{p} (1-|x_i|)^{-\beta} < \infty,$$

since each $x_i \in (-1, 1)$. Hence (4.1) holds.

B − M Assumption 2.2 states that the derivatives in (2.7) and (2.8) exist, which we have already shown, and that for sufficiently small $\varepsilon > 0$, for each x there exists $\kappa_x < \infty$ such that

$$\sup_{\|\theta\| \leqslant \varepsilon} \left| \frac{h_\theta(x)}{\|\theta\|^2} \right| \leqslant \kappa_x. \tag{4.2}$$

This result follows from (2.18), where in fact we have the stronger result that

$$\kappa = \sup_{x \in \mathscr{X}} \kappa_x < \infty. \tag{4.3}$$

B − M Assumption 2.3 is trivial in this problem since Θ is bounded. See the remark below Equation (2.5) in B − M. Thus the set $\mathscr{C}$ in B − M consists only of ϕ and $\mathscr{X}$, and hence can be ignored safely.

Now B − M Theorem 2.4 guarantees that an essentially complete class consists of all tests of the form (2.9), where

$$((\mu, M), H, c) \in \Xi - ((0, 0), 0, 0) \tag{4.4}$$

and

$$M_0 = M - \int_{\bar{\Theta} - \{0\}} \frac{\boldsymbol{\theta}\boldsymbol{\theta}'}{\|\theta\|^2} H(d\theta), \tag{4.5}$$

and

$$\Xi = \{((\mu, M), H, c) \mid (\mu, M) \in \Lambda(H), H \in \mathscr{F}(\bar{\Theta} - \{0\}), c \in \mathbb{R}\}. \tag{4.6}$$

(We take α in B − M large enough so that $\theta \in \bar{\Theta} \Rightarrow \|\theta\| < \alpha$.] The set $\Lambda(H)$ is a subset of $\mathbb{R}^p \times \mathscr{S}(p)$ defined in B − M (2.14). We will use B − M Example 4.6 to find $\Lambda(H)$, but first we reparametrize by letting

$$\pi = G\theta \in \mathscr{D}(p), \tag{4.7}$$

where G is the linear transformation from which $\pi_i = \theta_i - \theta_{i+1}$, $i = 1, \ldots, p-1$, and $\pi_p = \theta_p$. Then the transformed parameter space $G\Theta \equiv \Pi$ is locally one-sided, i.e., for some $\varepsilon > 0$,

$$\Pi \in B_\varepsilon = [[0, \infty)^{p-1} \times \mathbb{R}] \cap B_\varepsilon, \tag{4.8}$$

where B_ε is the ε-ball in $\mathscr{D}(p)$ around 0. From B – M Example 4.6 (with $K_1 = [0, \infty)^{p-1}$ and $q = 1$,) we have that if

$$\int \pi_i GH(d\pi) < \infty, \qquad i = 1, \ldots, p-1, \tag{4.9}$$

then

$$\begin{aligned} \Lambda(GH) = \{(\mu^*, M^*) \mid \mu^* \in C(\Pi) \text{ and } M_0^* \in \mathscr{D}(p), \\ m_i^* = 0,\ i = 1, \ldots, p-1,\ m_p^* \geqslant 0\}. \end{aligned} \tag{4.10}$$

If (4.9) fails, $\Lambda(GH)$ is empty. Here, GH is the measure induced on Π by G via (4.7). Now it can be seen from the definition of $\Lambda(H)$ in B – M that

$$\begin{aligned} \Lambda(H) &= \{(G^{-1}\mu^*, G^{-1}M^*(G')^{-1}) \mid (\mu^*, M^*) \in \Lambda(GH)] \\ &= \{(\mu, M) \mid \mu \in C(\Theta), M_0 = \gamma J, \gamma \geqslant 0\}. \end{aligned} \tag{4.11}$$

Hence (2.10) is equivalent to (4.4) via (4.11), proving that Φ is essentially complete for problem (1.8).

The verification of B – M Assumptions 2.1, 2.2, and 2.3 for problem (1.18) proceeds as for problem (1.8) since it shares $R_\theta(x)$ and has $\Theta^+ \subseteq \Theta$. Note that Θ^+ is locally pointed as in B – M Example 4.5. That is, there exists $a_0 \in \mathscr{D}(p)$ and $b_0 < 0$ such that for sufficiently small $\varepsilon > 0$,

$$\sup_{\|\theta\| \leqslant \varepsilon} \frac{\mathbf{a}_0' \boldsymbol{\theta}}{\|\theta\|} \leqslant b_0. \tag{4.12}$$

To see this, take $a_0 = I$, and note that by (1.19),

$$\sup_{\theta \in \Theta^+} \frac{\Sigma \theta_i}{\|\theta\|^2} = -1.$$

Thus B – M characterize the complete class as consisting of all tests of the form

$$\phi(x) = \left\{ {1 \atop 0} \right\} \qquad \text{if } \mu' l(x) + \int_{\Theta^+ - \{0\}} \frac{R_\theta(x) - 1}{\|\theta\|} H(d\theta) - c \left\{ {> \atop <} \right\} 0, \tag{4.13}$$

a.e. $[\nu]$, for some

$$(\mu, H, c) \in C(\Theta^+) \times \mathscr{F}(\bar{\Theta}^+ - \{0\}) \times \mathbb{R} - \{(0,0,0)\}. \tag{4.14}$$

But since $\mu' l(x) = -(\beta/p)\, \Sigma\mu_i \operatorname{tr} x$ as in (2.21), and $\mu \in C(\Theta^+)$ implies that $\Sigma\mu_i \leqslant 0$, we see that (4.13) and (4.14) are equivalent to (2.30) and (2.31). Hence Φ^* is essentially complete for problem (1.18).

Now turn to problem (1.29). The B – M Assumptions 2.1, 2.2, and 2.3 are fairly easy to verify by using the approach for the previous two problems, and by noting that

$$\begin{aligned}\inf_{\omega \in \Omega} \inf_{\Gamma \in O(p)} (1 + \operatorname{tr} y\Gamma\omega\Gamma') &= p \inf_{\lambda \in \Lambda} \inf_{\Gamma \in O(p)} (\operatorname{tr} y\Gamma\lambda\Gamma') \\ &= p \inf_{\lambda \in \Lambda} \Sigma y_i \lambda_{p-i+1} = p y_p > 0,\end{aligned} \tag{4.15}$$

so that $R^*_\omega(y)$ in (3.2) is finite. Since $l^*(x) \equiv 0$ (see (3.8), we can use B – M Remark 2.8 and Example 4.2 to show tha the class of tests of the form

$$\phi(x) = \{{}^1_0\} \qquad \text{as } \operatorname{tr} M_0 V^+(x) + \int_{\bar{\Omega} - \{0\}} \frac{R^*_\omega(x) - 1}{\|\omega\|^2} H(d\omega) - c\{\gtrless\}\, 0, \tag{4.16}$$

a.e. $[\nu^*]$, is essentialy complete, where

$$(M, H, c) \in \mathscr{S}(p) \times \mathscr{F}(\bar{\Omega} - \{0\}) \times \mathbb{R} - \{(0,0,0)\}. \tag{4.17}$$

Now (3.9) shows that their class is in fact Φ^* of (3.10), (3.11), and (3.12).

To complete the proofs of the theorems, we must show that the classes Φ, Φ^+, and Φ^* are minimal complete. These results follow from B – M Lemma 3.2, which requires verification of B – M Assumption 3.1. We will verify this assumption only for problem (1.8). The verification for the other problems can be dealt with similarly.

Consider problem (1.8). B – M Assumption 3.1 has four parts. Parts (i) and (iii) are trivial since $\mathscr{C} = \{\phi, \mathscr{X}\}$. Part (iv) requires that

$$\nu(\{x \mid d(x; \mu, M_0, H, c) = 0\}) = 0$$

for (μ, M_0, H, c) as in (2.10), which follows from the discussion after (2.29).

Part (ii) requires that for each $\phi \in \Phi$, there exists a sequence $\{J_i\} \subseteq \mathscr{F}(\Theta)$ such that

$$d_i(x) \equiv \int_{\Theta - \{0\}} R_\theta(x)\, J_i(d\theta) - J_i(\{0\}) \xrightarrow{i \to \infty} d(x) \qquad \text{for each } x, \tag{4.18}$$

where $d(x)$ is defined in (2.8), and

$$\lim_{i \to \infty} \int (\phi_i(x) - \phi(x))\, d_i(x)\, \nu(dx) = 0, \tag{4.19}$$

where

$$\phi_i(x) = \left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\} \qquad \text{as } d_i(x)\left\{\begin{smallmatrix}>\\<\end{smallmatrix}\right\} 0. \tag{4.20}$$

Now take $\phi \in \Phi$ and its attendant (μ, M_0, H, c), and define

$$\begin{aligned} \Theta_0 &= \{\theta \in \Theta \mid \|\theta\| \leqslant \tfrac{1}{10}\}, &\qquad \Theta_1 &= \bar{\Theta} - \Theta_0, \\ H_0(d\theta) &= H(d\theta)\, I_{\Theta_0}, & H_1(d\theta) &= H(d\theta)\, I_{\Theta_1}. \end{aligned} \tag{4.21}$$

Also, for $i \geqslant 1$, let $H_{1i} \in \mathscr{F}(\Theta - \{0\})$ be defined by

$$H_{1i}\left(\frac{i}{i+1} A\right) = \left(\frac{i}{i+1}\right)^2 H_1(A) \qquad \text{for } A \subseteq \Theta_1. \tag{4.22}$$

Then using the methods in B – M Lemma 2.5, we can find $\{J_i\}$ such that, from (4.18),

$$d_i(x) = A_i(x) + a_i(x), \tag{4.23}$$

where

$$A_i(x) = \mu_i' l(x) + \tfrac{1}{2} \operatorname{tr} M_i V(x) + \int_{\Theta_0 - \{0\}} \frac{h_\theta(x)}{\|\theta\|^2} H_{0i}(d\theta) - c_i, \tag{4.24}$$

with

$$\mu_i \to \mu, \qquad M_i \to M_0 + \int_{\Theta_0 - \{0\}} \frac{\boldsymbol{\theta}\boldsymbol{\theta}'}{\|\theta\|^2} H_0(d\theta), \qquad c_i \to c, \tag{4.25}$$

$$\int_{\Theta_0 - \{0\}} g(\theta)\, H_{0i}(d\theta) \to \int_{\Theta_0 - \{0\}} g(\theta)\, H_0(d\theta) \tag{4.26}$$

for any continuous bounded function g with $g(0) = 0$, and

$$a_i(x) = \int_{\Theta_1} \frac{R_\theta(x) - 1 - \boldsymbol{\theta}' l(x)}{\|\theta\|^2} H_{1i}(d\theta). \tag{4.27}$$

It is clear from (4.24) and (4.25) that

$$\begin{aligned} A_i(x) \to A(x) \equiv \mu' l(x) &+ \tfrac{1}{2} \operatorname{tr}\left(M_0 + \int_{\Theta_0 - \{0\}} \frac{\boldsymbol{\theta}\boldsymbol{\theta}'}{\|\theta\|^2} H_0(d\theta)\right) V(x) \\ &+ \int \frac{h_\theta(x)}{\|\theta\|^2} H_0(d\theta) - c. \end{aligned} \tag{4.28}$$

Now by (2.15) and (2.18) for $h_\theta(x)/\|\theta\|^2$, and by (2.19) and (2.20) for $l(x)$ and $V(x)$, we have that for some $N<\infty$,

$$|A(x)| \leqslant N \qquad \text{and} \qquad |A_i(x)| \leqslant N \qquad \text{for all } i, x. \tag{4.29}$$

Also, for a_i in (4.27), since $R_{b\theta}(x)=R_\theta(bx)$, by (4.22),

$$\begin{aligned} a_i(x) &= \int_\Theta \frac{R_\theta(x)-1-\boldsymbol{\theta}'l(x)}{\|\theta\|^2}\, H_{1i}(d\theta) \\ &= \int_{\Theta_1} \frac{R_{(i/i+1)\theta}(x)-1-(i/(i+1))\,\boldsymbol{\theta}'l(x)}{\|\theta\|^2}\, H_1(d\theta) \\ &= \int_{\Theta_1} \frac{R_\theta((i/(i+1))x)-1-\boldsymbol{\theta}'l((i/(i+1))x)}{\|\theta\|^2}\, H_1(d\theta) \\ &= a\left(\frac{i}{i+1}\,x\right), \end{aligned} \tag{4.30}$$

where

$$a(x)=\int_{\Theta_1} \frac{R_\theta(x)-1-\boldsymbol{\theta}'l(x)}{\|\theta\|^2}\, H_1(d\theta). \tag{4.31}$$

Since the integrand for $a((i/(i+1))x)$ is bounded in i for each fixed x and θ, and continuous in θ, we have that

$$a_i(x) \equiv a\left(\frac{i}{i+1}\,x\right) \to a(x). \tag{4.32}$$

Thus (4.23) through (4.27), (4.30), and (4.31) show that (4.18) holds, since $d(x)=A(x)+a(x)$.

Finally, note that $a(0)=0$, and since $a(x)$ is convex in x (see (2.30)), for $t>0$,

$$\begin{aligned} a(x)\leqslant t &\Rightarrow a\left(\frac{i}{i+1}\,x\right)\leqslant t \\ &\Rightarrow a_i(x)\leqslant t. \end{aligned} \tag{4.33}$$

Turn to (4.19). By (4.20), (4.23), and (4.29), when $a_i(x)>N$, $\phi_i(x)=1$, and by (4.28), (4.31), and (4.29), when $a(x)>N$, $\phi(x)=1$. Thus if $d_i(x)>2N$

then $a_i(x) > N$, hence by (4.33), $a(x) > N$, and $\phi_i(x) = \phi(x) = 1$ (a.e. [ν]). Thus

$$\begin{aligned}\lim_{i\to\infty} \int (\phi_i(x) - \phi(x))\, d_i(x)\, \nu(dx) \\ &= \lim_{i\to\infty} \int_{\{d_i(x) \leq 2N\}} (\phi_i(x) - \phi(x))\, d_i(x)\, \nu(dx) \\ &= 0, \end{aligned} \tag{4.34}$$

where the limit and integral can be interchanged by the bounded convergence theorem (the integrand is essentially nonnegative by definition of ϕ_i and d_i in (4.20)), and the limit of the integrand is zero a.e. [ν] by (2.9), (4.18), and (4.20). Thus (4.34) verifies (4.19), and the proof of Theorem 2.1 is complete.

References

[1] ANDERSON, T. W. (1984). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[2] BROWN, L. D., AND MARDEN, J. I. (1989). Complete class results for hypothesis testing problem with simple null hypotheses. *Ann. Statist.* **17**.

[3] FARRELL, R. H. (1976). *Techniques of Multivariate Calculation.* Springer-Verlag, New York/Berlin.

[4] GIRI, N. (1968). On tests of equality of two covariance matrices. *Ann. Math. Statist.* **39** 275–277.

[5] JAMES, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475–501.

[6] KIEFER, J., AND SCHWARTZ, R. (1965). Admissible Bayes character of T^2, R^2, and other fully invariant tests for classical multivariate normal problems. *Ann. Math. Statist.* **36** 747–770.

[7] MUIRHEAD, R. J. (1982). *Aspects of Multivariate Statistical Theory.* Wiley, New York.

[8] SUGIURA, N. Locally best invariant test for sphericity and the limiting distributions. *Ann. Math. Statist.* **43** 1312–1316.

[9] WIJSMAN, R. A. (1967). Cross-sections of orbits and their application to densities of maximal invariants. In *Proceedings, Fifth Berkeley Symp. Math. Statist. and Probab.* 1, pp. 389–400.

[10] MARSHALL, A. W. AND OLKIN, I. (1979). *Inequalities: Theory of Majorization and Its Applications.* Academic Press, New York.